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Quantum Effects of an Extra Compact Dimension on the Wave Function of the Universe

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We extended the direct quantum approach of the standard FRW cosmology from 4D to 5D and obtained a Hamiltonian formulation for a wave-like 5D FRW cosmology. Using a late-time approximation we isolated a y-part from the full wave function of the 5D Universe. Then we found that the compactness of the fifth dimension y yields a quantized spectrum for the momentum P_5 along the fifth dimension, and we have shown that the whole space-part of the wave function of the 5D Universe satisfies a 2D Schrödinger equation.

KEY WORDS: quantum cosmology; Hamiltonian formulation.

1. INTRODUCTION

Quantum cosmology has received extensive studies (Novello *et al.*, 1995; Wheeler *et al.*, 1964). Recently, a direct quantum approach was used to derive Hamiltonian formulation of the standard Friedmann–Robertson–Walker (FRW) cosmological models (Elbaz *et al.*, 1997; Novello *et al.*, 1996). The method is based on assumptions of validity of the classical FRW equations and a transformation from a pair of physical variables (ρ, θ) (the density of matter and the inverse of the Hubble radius), which describe the dynamics of a spatially homogeneous and isotropic perfect fluid, to another pair of canonical variables (q, p) (roughly the radius of the Universe and its rate of change). In this way, a canonical description of quantum FRW cosmology was obtained. In this paper, we have extended this procedure to a 5D FRW cosmology and have discussed the quantum effects of the fifth compact dimension on the wave function of the Universe.

The arrangement of this paper is as follows. In Section 2, we study a class of 5D wave-like cosmological solutions and derive Hamiltonian formulation of the 5D FRW cosmology. In Section 3, we use a late-time approximation to separate

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the 5D wave function of the Universe and isolate out the part of the fifth dimension. In Section 4, we study quantization of the wave function due to the compactness of the fifth dimension. Section 5 is a conclusion.

2. HAMILTONIAN FORMULATION OF 5D COSMOLOGY

We consider a FRW type 5D metric

$$dS^2 = -B^2(t, y)\, dt^2 + A^2(t, y)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) + C^2(t, y)\, dy^2, \qquad (2.1)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\ d\varphi^2$ and the coordinates $x^A = (t, r\theta\varphi, y)$ (here and throughout this paper, lowercase Greek letters run 0, 123 and uppercase Latin letters run 0, 123, 5). Using the 4D part of the 5D metric (2.1) we can calculate all the nonvanishing components of the 4D Einstein tensor ${}^{(4)}G_{\alpha\beta}$, which are as follows:

$${}^{(4)}G^0_0 = 3\frac{\dot{A}^2}{A^2B^2} + \frac{3k}{A^2}, \qquad (2.2)$$

$${}^{(4)}G^1_1 = {}^{(4)}G^2_2 = {}^{(4)}G^3_3 = -2\frac{\ddot{A}}{AB^2} - \frac{\dot{A}^2}{A^2B^2} - \frac{k}{A^2} + 2\frac{\dot{A}\dot{B}}{AB^3}. \qquad (2.3)$$

where a dot denotes partial derivative with respect to time t.

It is known that solutions which are empty in 5D may have matter in 4D (Overduin and Wesson, 1997; Wesson, 1999). Therefore, one can define an induced 4D energy-momentum tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} \equiv {}^{(4)}G_{\alpha\beta}. \qquad (2.4)$$

It is found that this $T_{\alpha\beta}$ can take the form of a perfect fluid,

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + Pg_{\alpha\beta}, \qquad (2.5)$$

where ρ and P are the energy density and the pressure of the induced matter, respectively.

Now we let the equation of state be

$$P = \gamma\rho, \qquad (2.6)$$

where $\gamma = 0$ and $\gamma = 1/3$ represent matter-dominated and radiation-dominated eras, respectively. Then substituting (2.2), (2.3), (2.5), and (2.6) in (2.4), we obtain the well-known Friedmann equation

$$3\frac{\dot{A}^2}{A^2} = B^2\rho - \frac{3kB^2}{A^2}, \qquad (2.7)$$

and the Raychaudhuri equation

$$3\frac{\ddot{A}}{A} - 3\frac{\dot{A}\dot{B}}{AB} + \frac{B^2}{2}(1+3\gamma)\rho = 0. \tag{2.8}$$

The conservation law for the energy-momentum $T_{\alpha\beta}$ gives us another equation

$$\dot{\rho} + 3(1+\gamma)\frac{\dot{A}}{A}\rho = 0. \tag{2.9}$$

An exact 5D cosmological solution satisfying the empty 5D equations $R_{AB} = 0$ is (Liu and Wesson, 1994)

$$dS^2 = -B^2(u)\,dt^2 + A^2(u)(dr^2 + r^2 d\Omega^2) + B^2(u)\,dy^2, \tag{2.10}$$

where k is chosen to be zero,

$$A(u) = (hu)^{\frac{1}{2+3\gamma}},$$
$$B(u) = (hu)^{-\frac{1+3\gamma}{2(2+3\gamma)}}, \tag{2.11}$$

and $u \equiv t - y$. The energy density ρ and pressure P are

$$\rho = \frac{3h^2}{(2+3\gamma)^2}A^{-3(1+\gamma)}, \qquad P = \gamma\rho. \tag{2.12}$$

This is a wave-like cosmological solution which can be interpreted as a shock wave propagating along the fifth dimension (Wesson *et al.*, 2000).

Now let us return to Eqs. (2.7)–(2.9). There are three basic equations from which we want to derive the Hamiltonian formulation. Note that since $A = A(u)$ and $u = t - y$, we have $\dot{A} = \partial A/\partial t = dA/du$. Therefore, by using a transformation

$$\tilde{u} = \int B(u)\,du, \tag{2.13}$$

we can reduce Eqs. (2.7)–(2.9) to

$$\frac{3\overset{*}{A}{}^2}{A^2} = \rho, \tag{2.14}$$

$$\frac{3\overset{**}{A}}{A} + \frac{1+3\gamma}{2}\rho = 0, \tag{2.15}$$

$$\overset{*}{\rho} + 3(1+\gamma)\frac{\overset{*}{A}}{A}\rho = 0, \tag{2.16}$$

where an asterisk denotes derivative with respect to $\tilde{u}$. These three Eqs. (2.14)–(2.16) are of the same forms as in the standard 4D FRW solutions algebraically.

Therefore, we can follow the procedure given by Elbaz *et al.* (1997; see also Novello *et al.*, 1996) to derive the canonical formulation of the cosmology.

First, we introduce an expansion parameter Θ, $\Theta \equiv 3\frac{\overset{*}{A}}{A}$, which brings (2.14)–(2.16) to

$$\frac{1}{3}\Theta^2 = \rho, \tag{2.17}$$

$$\overset{*}{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(1+3\gamma)\rho = 0, \tag{2.18}$$

$$\overset{*}{\rho} + (1+\gamma)\rho\Theta = 0. \tag{2.19}$$

Then we choose a new set of canonical variables (q, p) with

$$\begin{aligned} q &= b\rho^{-1/(3+3\gamma)}, \\ p &= \frac{1}{3}b\Theta\rho^{-1/(3+3\gamma)}, \end{aligned} \tag{2.20}$$

where b is an arbitrary constant. In this way, we arrive at the following Hamiltonian

$$\mathcal{H}(q, p) = \frac{1}{2}p^2 - \frac{1}{6}b^{3(1+\gamma)}q^{-(1+3\gamma)}. \tag{2.21}$$

This Hamiltonian describes a particle with momentum p in a potential $V(q) = -\frac{1}{6}b^{3(1+\gamma)}q^{-(1+3\gamma)}$. Using relations (2.17)–(2.21) we can verify that the two Hamilton equations $\overset{*}{q} = \partial\mathcal{H}/\partial p$ and $\overset{*}{p} = -\partial\mathcal{H}/\partial q$ hold. Note that here the time coordinate is not t but $\tilde{u}$, that is, $\overset{*}{q} \equiv dq/d\tilde{u}$ and $\overset{*}{p} \equiv dp/d\tilde{u}$. This will make a difference between the two canonical formulations of the 4D and 5D cosmologies as shown in the next section.

3. WAVE FUNCTION AND LATE-TIME APPROXIMATION

From the Hamiltonian (2.21) we can employ the standard quantization procedure to write the corresponding Schrödinger equation. Because the time coordinate in the Hamiltonian formulation is $\tilde{u}$, so the time-dependent wave function is of the form

$$\Psi(q, \tilde{u}) = e^{-i\bar{\tilde{E}}\tilde{u}}\varphi(q). \tag{3.1}$$

The correspondence principle $p \to -i\frac{\partial}{\partial q}$ gives a stationary Schrödinger equation for the stationary wave function $\varphi(q)$ as

$$\frac{1}{2}\frac{\partial^2}{\partial q^2}\varphi(q) + \left[\tilde{E} + \frac{1}{6}b^{3(1+\gamma)}q^{-(1+3\gamma)}\right]\varphi(q) = 0. \tag{3.2}$$

In (3.1) and (3.2) $\tilde{E}$ is the associated eigenvalue of the operator $\hat{\mathcal{H}}$, $\tilde{E} = \langle\Psi(q, \tilde{u})|\hat{\mathcal{H}}|\Psi(q, \tilde{u})\rangle$. Quantized bound states with negative $\tilde{E}$ of the Schrödinger Eq. (3.2) were studied by Novello *et al.* (1996) and Mongan (1999, 2000, 2001). Their results can also be used here.

Since $\tilde{u}$ is not the proper time of the cosmic fluid, we are not sure if we can interpret $\tilde{E}$ as the energy. Therefore, we look for the relation between $\tilde{u}$ and the proper time. Now let us consider the 5D metric (2.10), from which we see that the 2D line-element in the $t - y$ plane can be written as

$$ds^2 = -B^2(u)\, du\, d(t + y). \tag{3.3}$$

Thus by defining

$$U \equiv \int B^2(u)\, du, \qquad V \equiv (t + y). \tag{3.4}$$

we get $ds^2 = -dU\, dV$. Then let

$$U \equiv T - \lambda Y, \qquad V \equiv T + \lambda Y, \tag{3.5}$$

where λ is a constant to be determined later, we get $ds^2 = -dT^2 + \lambda^2\, dY^2$. Therefore we find that T is the proper time. Substituting (2.11) in (2.13) and (3.4), we find

$$h\tilde{u} = \frac{2(2 + 3\gamma)}{3(1 + \gamma)}(hu)^{\frac{3(1+\gamma)}{2(2+3\gamma)}}, \tag{3.6}$$

$$hU = (2 + 3\gamma)(hu)^{\frac{1}{2+3\gamma}}. \tag{3.7}$$

From these two equations we obtain

$$\tilde{u} = W_\gamma U^{\frac{3(1+\gamma)}{2}} = W_\gamma (T - \lambda Y)^{\frac{3(1+\gamma)}{2}}, \tag{3.8}$$

where

$$W_\gamma \equiv \frac{2}{3(1 + \gamma)}\left(\frac{h}{2 + 3\lambda}\right)^{\frac{1+3\gamma}{2}}. \tag{3.9}$$

Now let us consider a short period in a later time of the Universe. This means that we choose $T = 0$ as the beginning of the Universe (on the $Y = 0$ hypersurface) and write $T = T_0 + \tau$ with $|\tau| \ll T_0$. Then (3.8) gives

$$\begin{aligned}\tilde{u} &= W_\gamma T_0^{\frac{3(1+\gamma)}{2}}\left(1 + \frac{\tau - \lambda Y}{T_0}\right)^{\frac{3(1+\gamma)}{2}} \\ &\approx W_\gamma T_0^{\frac{3(1+\gamma)}{2}} + \frac{3(1 + \gamma)}{2} T_0^{\frac{1+3\gamma}{2}}(\tau - \lambda Y).\end{aligned} \tag{3.10}$$

Using this relation we find that, up to a constant factor, the wave function $\Psi(q, \tilde{u})$ in (3.1) becomes

$$\Psi(q, Y, \tau) \approx e^{-iE\tau}\ e^{iP_5 Y}\varphi(q). \tag{3.11}$$

where

$$E \equiv \chi \tilde{E},$$
$$P_5 \equiv \chi \tilde{E}\lambda, \tag{3.12}$$

and

$$\chi = \frac{3(1+\gamma)W_\gamma}{2} T_0^{(1+3\gamma)/2} = \left(\frac{hT_0}{2+3\gamma}\right)^{\frac{1+3\gamma}{2}}. \tag{3.13}$$

Thus we have successfully obtained an approximate wave function $\Psi(q, Y, \tau)$ as shown in (3.11), in which the three variables τ, q, and Y were separated.

4. QUANTIZATION WITH A COMPACT FIFTH DIMENSION

The wave function $\Psi(q, Y, \tau)$ in (3.11) is in a form with three separated variables τ, q, and Y. Denote the Y-part of Ψ as $\Phi(Y)$, $\Phi(Y) = e^{iP_5 Y}$, then the momentum operator $\hat{P}_5 = -i\frac{\partial}{\partial Y}$ gives

$$\hat{P}_5 \Phi = P_5 \Phi. \tag{4.1}$$

So P_5 is the eigenvalue of the momentum along the fifth dimension.

Now we suppose the fifth dimension to be a circle with a radius R; i.e., $Y = R\phi$. So we have $\Phi = \Phi(\phi) = e^{iP_5 R\phi}$. The boundary condition requires that $\Phi(\phi)$ must be periodic with period 2π. It follows that we must have

$$P_5 = \frac{n}{R}, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{4.2}$$

In this way, the fifth momentum P_5 and the corresponding wave function $\Phi(\phi)$ are quantized.

The momentum operator $\hat{P}_5 = -i\frac{\partial}{\partial Y}$, acting on Eq. (4.1) from the left hand side, gives

$$\frac{1}{2}\frac{\partial^2}{\partial Y^2}\Phi(Y) + \frac{n^2}{2R^2}\Phi(Y) = 0. \tag{4.3}$$

Let $\psi(q, Y) \equiv \varphi(q)\Phi(Y)$, then Eqs. (3.2) and (4.3) give

$$\frac{1}{2}\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial Y^2}\right)\psi(q, Y) + \left(\tilde{E} + \frac{n^2}{2R^2} + \frac{1}{6}b^{3(1+\gamma)}q^{-(1+3\gamma)}\right)\psi(q, Y) = 0. \tag{4.4}$$

So we obtain a stationary 2D Schrödinger equation.

5. CONCLUSION

In this paper we have obtained a Hamiltonian formulation for a 5D cosmology. This Hamiltonian leads immediately to a stationary 1D Schrödinger Eq. (3.2). Thus the corresponding full wave function can describe quantized states of a 5D Universe. By using a late-time approximation we have successfully separated the wave function into three parts, $\Psi(q, Y, \tau) \approx e^{-iE\tau}\ e^{iP_5 Y}\varphi(q)$, corresponding to the proper time τ, the variable q (which describes the radius of the Universe), and the fifth coordinate Y, respectively. Notice that this wave function just valid in a later time of the Universe with $|\tau| \ll T_0$, where T_0 is the age of the Universe at that time. Then by assuming the compact fifth dimension to be a circle, we obtain a quantized spectrum for the momentum P_5 in the fifth direction. Note that the q-part of the wave function, $\varphi(q)$, satisfies the 1D Schrödinger Eq. (3.2), for which bound states were given (Novello *et al.*, 1996). Using their results, as well as the spectrum for P_5, we can obtain bound states for the whole space-part of the wave function, $\Psi(q, Y) = e^{iP_5 Y}\varphi(q)$, which satisfies the 2D Schrödinger Eq. (4.4). Further studies are needed on this.

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